

MULTISCALE EXPANSION OF THE LATTICE POTENTIAL KDV EQUATION ON FUNCTIONS OF INFINITE SLOW-VARYNESS ORDER

R. HERNANDEZ HEREDERO, D. LEVI, M. PETRERA, AND C. SCIMITERNA

ABSTRACT. We present a discrete multiscale expansion of the lattice potential Korteweg-de Vries (lpKdV) equation on functions of infinite order of slow-varyness. To do so we introduce a formal expansion of the shift operator on many lattices holding at all orders. The lowest secularity condition from the expansion of the lpKdV equation gives a nonlinear lattice equation, depending on shifts of all orders, of the form of the nonlinear Schrödinger (NLS) equation.

1. INTRODUCTION

Reductive perturbation techniques [14, 15] have proved to be important tools to find approximate solutions for many physical problems by reducing a given nonlinear partial differential equation to a simpler equation, which is often integrable [4]. Recently a few attempts to carry over this approach to partial difference equations have been proposed [3, 6–9].

The basic tool of the discrete reductive perturbation technique developed in [7, 9] is a proper multiscale expansion carried out by introducing various scales and lattices defined on them. Let us recall some of the main results. The transformation between two different lattices of indices n and n_1 is given by [5]

$$\Delta_n^j u_n \doteq \sum_{i=0}^j (-1)^{j-i} \binom{j}{i} u_{n+i} = j! \sum_{i=j}^{\infty} \frac{P_{i,j}}{i!} \Delta_{n_1}^i u_{n_1}, \quad (1.1)$$

where $u_n : \mathbb{Z} \rightarrow \mathbb{R}$ is a function defined on a lattice of index $n \in \mathbb{Z}$ and $u_{n_1} : \mathbb{Z} \rightarrow \mathbb{R}$ is the same function on a lattice of index $n_1 \in \mathbb{Z}$. By the symbol Δ we mean the standard forward difference of the function u w.r.t. its subscript, e.g. $\Delta_n u_n \doteq u_{n+1} - u_n$. Here the coefficients $P_{i,j}$ are given by

$$P_{i,j} \doteq \sum_{k=j}^i \omega^k \mathcal{S}_i^k \mathfrak{S}_k^j,$$

where ω is the ratio of the increment in the lattice of variable n_1 with respect to that of variable n . The coefficients \mathcal{S}_i^k and \mathfrak{S}_k^j are the Stirling numbers of the first and second kind respectively [2]. Eq. (1.1) implies that a finite difference in the discrete variable n depends on an *infinite* number of differences on the variable n_1 , e.g. the function u_{n+1} can be written as a combination of the functions u_i 's for i varying on an infinite subset of the lattice with index n_1 . Formula (1.1) can be inverted yielding

$$\Delta_{n_1}^j u_{n_1} = j! \sum_{i=j}^{\infty} \frac{Q_{i,j}}{i!} \Delta_n^i u_n, \quad Q_{i,j} \doteq \sum_{k=j}^i \omega^{-k} \mathcal{S}_i^k \mathfrak{S}_k^j.$$

In [7, 9] one has introduced the notion of *slow-varyness of order ℓ* for a function u_n iff $\Delta_n^{\ell+1} u_n = 0$ (or equivalently $\Delta_{n_1}^{\ell+1} u_{n_1} = 0$, see [9] for further details). This definition enables to reduce infinite series to a finite number of terms. Moreover one has considered a generalization of formula (1.1) in order to deal with functions $u_n = u_{n; \{n_i\}_{i=1}^K}$ depending on a finite number K of lattice variables n_i , $1 \leq i \leq K$. The computations done in [7] proved that an integrable

lattice equation, as the lattice potential KdV (lpKdV) equation, reduces to a completely discrete nonlinear Schrödinger (dNLS) equation of the form

$$i(\phi_{n,m+1} - \phi_{n,m}) + c_1(\phi_{n+1,m} - 2\phi_{n,m} + \phi_{n-1,m}) + c_2\phi_{n,m}|\phi_{n,m}|^2 = 0, \quad (1.2)$$

being c_1, c_2 two real coefficients. Let us stress the fact that the above dNLS equation has a completely local nonlinear part. We refer to the papers [7, 9] for further details. It has been proved by singularity confinement [13] and algebraic entropy [16] that the constructed dNLS equation is not integrable.

In [4] Calogero and Eckhaus have shown that a necessary condition for the integrability of a nonlinear partial differential equation is that its multiscale reduction be integrable. Here, trying to find a lattice analogue of the Calogero-Eckhaus theorem, we extend the techniques developed in [7, 9] to any order of slow-varyness.

In Section 2 we introduce a formal multiscale expansion holding at all orders of slow-varyness. Then in Section 3 we apply this technique to the lpKdV equation, thus providing an extension to all orders of the dNLS equation obtained in [7, 9]. Finally, Section 4 is devoted to concluding remarks.

2. MULTISCALE EXPANSION ON A LATTICE

2.1. Lattices and shifts defined on them. Let $u_n : \mathbb{Z} \rightarrow \mathbb{R}$ be a function defined on a lattice of index $n \in \mathbb{Z}$. We can always extend it to a real function $u : \mathbb{R} \rightarrow \mathbb{R}$ by defining a real continuous variable $x \doteq n\sigma_x$, where $\sigma_x \in \mathbb{R}$ is the lattice spacing.

Let us define the shift operator T_n such that $T_n u_n \doteq u_{n+1}$. For $u(x)$ we can introduce the operator T_x , corresponding to T_n , such that $T_x u(x) \doteq u(x + \sigma_x)$. The Taylor expansion of $u(x + \sigma_x)$ centered in x reads

$$T_x u(x) = u(x) + \sigma_x u^{(1)}(x) + \frac{\sigma_x^2}{2} u^{(2)}(x) + \dots + \frac{\sigma_x^i}{i!} u^{(i)}(x) + \dots = \sum_{i=0}^{\infty} \frac{\sigma_x^i}{i!} u^{(i)}(x), \quad (2.1)$$

where $u^{(i)}(x) \doteq d^i u(x)/dx^i \doteq d_x^i u(x)$, being d_x the total derivative operator. Eq. (2.1) suggests the following formal expansion for the differential operator T_x :

$$T_x \doteq e^{\sigma_x d_x} = \sum_{i=0}^{\infty} \frac{\sigma_x^i}{i!} d_x^i.$$

Introducing a formal derivative with respect to the index n , say δ_n , we can define, by analogy with T_x , the operator T_n as

$$T_n \doteq e^{\delta_n} = \sum_{i=0}^{\infty} \frac{\delta_n^i}{i!}. \quad (2.2)$$

The formal expansion (2.2) can be inverted, yielding

$$\delta_n = \ln T_n = \ln(1 + \Delta_n) = \sum_{i=1}^{\infty} \frac{(-1)^{i-1}}{i} \Delta_n^i, \quad (2.3)$$

where $\Delta_n \doteq \Delta_n^+ \doteq T_n - 1$ is the discrete first right difference operator w.r.t. the variable n , see eq. (1.1). Notice that this is just one of the possible inversion formulas for the operator δ_n . For example it can be also written in terms of left difference operators $\Delta_n^- \doteq 1 - T_n^{-1}$:

$$\delta_n = -\ln T_n^{-1} = -\ln(1 - \Delta_n^-) = \sum_{i=1}^{\infty} \frac{(\Delta_n^-)^i}{i},$$

or in terms of symmetric difference operators $\Delta_n^s \doteq (T_n - T_n^{-1})/2$:

$$\delta_n = \sinh^{-1} \Delta_n^s = \sum_{i=1}^{\infty} \frac{P_{i-1}(0)}{i} (\Delta_n^s)^i,$$

where $P_i(x)$ is the i -th Legendre polynomial evaluated in $x = 0$. Hence the δ_n operators are formal series containing infinite powers of Δ_n , but, acting on slow-varying functions of order ℓ , they reduce to polynomials in Δ_n of order at most ℓ . Consequently, any formula written in terms of powers of δ_n for a given slow-varyness order ℓ , contains its version for smaller orders $j < \ell$.

The convergence of the series $\delta_n u_n$ depends on the analyticity properties of the function u_n . Hence, from now on, we will proceed formally, considering that the developments can be justified a posteriori, because the functions which finally appear (solutions to the resulting difference equations) will have the correct analyticity properties.

2.2. Dilations on the lattice. Let us introduce a second lattice, obtained from the first one by a dilation. At first it is convenient to visualize the problem as a change of variable between the continuous variable $x \in \mathbb{R}$ and a new continuous variable $x_1 \doteq \epsilon x$, $0 < \epsilon \ll 1$. On the lattice one considers a change from the index $n \doteq x/\sigma_x$ to the new index $n_1 \doteq x_1/\sigma_{x_1}$, where σ_{x_1} is the new spacing. Assuming that $\sigma_{x_1} \gg \sigma_x$ we can set $\sigma_x \doteq \epsilon \sigma_{x_1}$, $0 < \epsilon \ll 1$, so that $n_1 = \epsilon n$. As $n, n_1 \in \mathbb{Z}$, $\epsilon \epsilon$ is a rational number and one can define in all generality $\epsilon \epsilon \doteq M_1/N \ll 1$ with $M_1, N \in \mathbb{N}$. However, if we want that the lattice of index n_1 is a sublattice of the lattice of index n , we have also to require that $M_1/N = 1/M$ with $M \in \mathbb{N}$.

The relationship between the discrete derivatives defined in the two lattices is given by eq. (1.1), for which we shall give a straightforward proof based on the well-known formulas [2]

$$(e^x - 1)^j = j! \sum_{k=j}^{\infty} \frac{\mathfrak{S}_k^j}{k!} x^k, \quad [\ln(1+x)]^k = k! \sum_{i=k}^{\infty} \frac{\mathcal{S}_i^k}{i!} x^i.$$

Here $n_1 = n(M_1/N)$ and thus $\omega = M_1/N$. From the expression of Δ_n^j , $j \in \mathbb{N}$, we get by a straightforward algebra:

$$\begin{aligned} \Delta_n^j &= (T_n - 1)^j = (e^{\sigma_x d_x} - 1)^j = j! \sum_{k=j}^{\infty} \mathfrak{S}_k^j \frac{(\sigma_x d_x)^k}{k!} = \\ &= j! \sum_{k=j}^{\infty} \frac{\mathfrak{S}_k^j}{k!} \left(\frac{\epsilon \sigma_x}{\sigma_{x_1}} \right)^k (\sigma_{x_1} d_{x_1})^k = j! \sum_{k=j}^{\infty} \frac{\mathfrak{S}_k^j}{k!} \left(\frac{\epsilon \sigma_x}{\sigma_{x_1}} \right)^k [\ln(1 + \Delta_{n_1})]^k = \\ &= j! \sum_{k=j}^{\infty} \frac{\mathfrak{S}_k^j}{k!} \left(\frac{\epsilon \sigma_x}{\sigma_{x_1}} \right)^k k! \sum_{i=k}^{\infty} \frac{\mathcal{S}_i^k}{i!} \Delta_{n_1}^i = j! \sum_{i=j}^{\infty} \frac{1}{i!} \left[\sum_{k=j}^i \left(\frac{\epsilon \sigma_x}{\sigma_{x_1}} \right)^k \mathcal{S}_i^k \mathfrak{S}_k^j \right] \Delta_{n_1}^i. \end{aligned}$$

As $\sigma_x = \epsilon \sigma_{x_1}$ and $\epsilon \epsilon = M_1/N$ eq. (1.1) is proven.

2.3. Discrete multiscale expansion. We present here the formulas necessary to construct a discrete multiscale expansion. According to the definitions given above, let us consider $u \doteq u_{n;n_1} = u(x; x_1)$ as a function depending on a fast index n and a slow index $n_1 = n(M_1/N)$. At the continuous level, the total derivative d_x acting on functions $u(x; x_1)$ is the sum of partial derivatives, i.e. $d_x = \partial_x + \epsilon \partial_{x_1}$. What is the situation on the lattice? Let us construct a relation between the total shift operator T_n and the partial shift operators $\mathcal{T}_n, \mathcal{T}_{n_1}$ ($\mathcal{T}_n u_{n;n_1} = u_{n+1;n_1}, \mathcal{T}_{n_1} u_{n;n_1} = u_{n;n_1+1}$). As

$$T_x = e^{\sigma_x d_x} = e^{\sigma_x \partial_x} e^{\epsilon \sigma_x \partial_{x_1}},$$

we can write

$$T_n = e^{\delta_n} e^{(M_1/N) \delta_{n_1}} \doteq \mathcal{T}_n \mathcal{T}_{n_1}^{(M_1/N)}, \quad (2.4)$$

with

$$\mathcal{T}_n \doteq \sum_{i=0}^{\infty} \frac{\delta_n^i}{i!}, \quad \mathcal{T}_{n_1}^{(M_1/N)} \doteq \sum_{i=0}^{\infty} \frac{(M_1/N)^i}{i!} \delta_{n_1}^i, \quad (2.5)$$

where δ_{n_1} is given by eq. (2.3) with n substituted by n_1 .

Eq. (2.4) can be easily extended to the case of K slow variables $x_i \doteq \epsilon^i x$, $1 \leq i \leq K$. The action of the shift operator T_n on a function $u \doteq u_{n; \{n_i\}_{i=1}^K}$ depending on both fast and slow variables can be written in terms of the partial shifts $\mathcal{T}_n, \mathcal{T}_{n_i}$:

$$T_n \doteq \mathcal{T}_n \prod_{i=1}^K \mathcal{T}_{n_i}^{(\epsilon_{n_i})}, \quad (2.6)$$

where the ϵ_{n_i} 's are suitable functions of ϵ and ε depending parametrically on some integer coefficients $M_i \in \mathbb{N}$, $1 \leq i \leq K$.

To develop the fields appearing in partial difference equations with two independent discrete variables one has to consider the action of the operator (2.6) on a function depending on two fast indices n and m , and on a set of $K_n + K_m$ slow variables $\{n_i\}_{i=1}^{K_n}$ and $\{m_i\}_{i=1}^{K_m}$, i.e. on $u \doteq u_{n,m; \{n_i\}_{i=1}^{K_n}, \{m_i\}_{i=1}^{K_m}}$. Notice that in principle it is possible to consider $K_n = K_m = \infty$. For the moment we assume a common definition of the small parameter ϵ for both discrete variables n and m , but we denote with M_i the integers for the slow variables n_i and with \widetilde{M}_i the ones for m_i . We have:

$$\epsilon_{n_i} \doteq \frac{M_i}{N^i}, \quad 1 \leq i \leq K_n, \quad \epsilon_{m_i} \doteq \frac{\widetilde{M}_i}{N^i}, \quad 1 \leq i \leq K_m.$$

In the next Section we shall consider a partial difference equation defined on a quadrilateral lattice, namely an equation of the type $f(u, T_n u, T_m u, T_n T_m u) = 0$. Assuming $K_n = 1$ and $K_m = K$ we get from eqs. (2.5, 2.6) the following expansions for the shift operators appearing in the equation $f(u, T_n u, T_m u, T_n T_m u) = 0$:

$$T_n = \mathcal{T}_n \mathcal{T}_{n_1}^{(\epsilon_{n_1})} = \mathcal{T}_n \left[1 + \frac{1}{N} M_1 \delta_{n_1} + \frac{1}{2} \frac{M_1^2}{N^2} \delta_{n_1}^2 + \frac{1}{6} \frac{M_1^3}{N^3} \delta_{n_1}^3 + O(1/N^4) \right], \quad (2.7)$$

$$\begin{aligned} T_m &= \mathcal{T}_m \prod_{i=1}^K \mathcal{T}_{m_i}^{(\epsilon_{m_i})} = \\ &= \mathcal{T}_m \left[1 + \frac{1}{N} \widetilde{M}_1 \delta_{m_1} + \frac{1}{N^2} \left(\frac{\widetilde{M}_1^2}{2} \delta_{m_1}^2 + \widetilde{M}_2 \delta_{m_2} \right) + \right. \\ &\quad \left. + \frac{1}{N^3} \left(\frac{\widetilde{M}_1^3}{6} \delta_{m_1}^3 + \widetilde{M}_1 \widetilde{M}_2 \delta_{m_1} \delta_{m_2} + \widetilde{M}_3 \delta_{m_3} \right) + O(1/N^4) \right], \end{aligned} \quad (2.8)$$

$$\begin{aligned} T_n T_m &= \mathcal{T}_n \mathcal{T}_{n_1}^{(\epsilon_{n_1})} \mathcal{T}_m \prod_{i=1}^K \mathcal{T}_{m_i}^{(\epsilon_{m_i})} = \\ &= \mathcal{T}_n \mathcal{T}_m \left[1 + \frac{1}{N} (M_1 \delta_{n_1} + \widetilde{M}_1 \delta_{m_1}) + \right. \\ &\quad \left. + \frac{1}{N^2} \left(\frac{M_1^2}{2} \delta_{n_1}^2 + M_1 \widetilde{M}_1 \delta_{n_1} \delta_{m_1} + \frac{\widetilde{M}_1^2}{2} \delta_{m_1}^2 + \widetilde{M}_2 \delta_{m_2} \right) + \right. \\ &\quad \left. + \frac{1}{N^3} \left(\frac{M_1^3}{6} \delta_{n_1}^3 + \frac{M_1^2}{2} \widetilde{M}_1 \delta_{n_1}^2 \delta_{m_1} + \frac{\widetilde{M}_1^2}{2} M_1 \delta_{n_1} \delta_{m_1}^2 + M_1 \widetilde{M}_2 \delta_{n_1} \delta_{m_2} + \right. \right. \\ &\quad \left. \left. + \frac{\widetilde{M}_1^3}{6} \delta_{m_1}^3 + \widetilde{M}_1 \widetilde{M}_2 \delta_{m_1} \delta_{m_2} + \widetilde{M}_3 \delta_{m_3} \right) + O(1/N^4) \right]. \end{aligned} \quad (2.9)$$

3. MULTISCALE EXPANSION OF THE LPKdV EQUATION

The lattice potential Korteweg-de Vries (lpKdV) equation is given by [12]:

$$\mathcal{P} \doteq (p - q + u_{n,m+1} - u_{n+1,m})(p + q - u_{n+1,m+1} + u_{n,m}) - (p^2 - q^2) = 0, \quad (3.1)$$

where p, q , $p \neq q$, are two real parameters. The above equation is probably the best-known completely discrete nonlinear equation which involves just four points which lay on two orthogonal infinite lattices and it is nothing else but the nonlinear superposition formula for the Korteweg-de Vries equation.

By defining $\mu \doteq p - q$ and $\zeta \doteq p + q$, eq. (3.1) can be written as

$$\mathcal{P} \doteq [\mu(T_n T_m u - u) + \zeta(T_n u - T_m u)] - [(T_n u - T_m u)(T_n T_m u - u)] \doteq \mathcal{P}_\ell - \mathcal{P}_{n\ell} = 0, \quad (3.2)$$

where \mathcal{P}_ℓ and $\mathcal{P}_{n\ell}$ denote respectively the linear and the nonlinear part of the lpKdV equation. The linear part \mathcal{P}_ℓ has a travelling wave solution of the form $u = \exp \{i[\kappa n - \omega(\kappa)m]\}$ with

$$\omega(\kappa) = -2 \arctan \left(\frac{\zeta + \mu}{\zeta - \mu} \tan \frac{\kappa}{2} \right). \quad (3.3)$$

Applying the expansions (2.7, 2.8, 2.9) to the function $u \doteq u_{n,m;n_1,\{m_i\}_{i=1}^K}$, \mathcal{P}_ℓ takes the form

$$\mathcal{P}_\ell = \sum_{i=0}^{\infty} \frac{1}{N^i} L_i u,$$

where the operators $L_i \doteq L_i(\mathcal{T}_n, \mathcal{T}_m, \delta_{n_1}, \{\delta_{m_j}\}_{j=1}^K)$ can be constructed in a recursive way. The lowest operators L_i , $i = 0, 1, 2$, read

$$\begin{aligned} L_0 &\doteq \mu(\mathcal{T}_n \mathcal{T}_m - 1) + \zeta(\mathcal{T}_n - \mathcal{T}_m), \\ L_1 &\doteq \mu \mathcal{T}_n \mathcal{T}_m \left(M_1 \delta_{n_1} + \widetilde{M}_1 \delta_{m_1} \right) + \zeta \left(M_1 \mathcal{T}_n \delta_{n_1} - \widetilde{M}_1 \mathcal{T}_m \delta_{m_1} \right), \\ L_2 &\doteq \mu \mathcal{T}_n \mathcal{T}_m \left[\frac{M_1^2}{2} \delta_{n_1}^2 + M_1 \widetilde{M}_1 \delta_{n_1} \delta_{m_1} + \frac{\widetilde{M}_1^2}{2} \delta_{m_1}^2 + \widetilde{M}_2 \delta_{m_2} \right] + \\ &\quad + \zeta \left[\frac{M_1^2}{2} \mathcal{T}_n \delta_{n_1}^2 - \mathcal{T}_m \left(\frac{\widetilde{M}_1^2}{2} \delta_{m_1}^2 + \widetilde{M}_2 \delta_{m_2} \right) \right]. \end{aligned}$$

As nonlinearity generate harmonics, let us now expand the function u as

$$u \doteq \sum_{\alpha \in \mathbb{Z}} u^{(\alpha)}(n_1, \{m_i\}_{i=1}^K; N) e^{i\alpha(\kappa n - \omega m)}.$$

As u is assumed to be real then $u^{(-\alpha)} = \bar{u}^{(\alpha)}$, where by \bar{u} we denote the complex conjugate of u . Moreover, if the nonlinear part should enter as a perturbation in the multiscale expansion of eq. (3.2), we need $u^{(\alpha)}(n_1, \{m_i\}_{i=1}^K; \infty) = 0$, $\forall \alpha \in \mathbb{Z}$. This implies that we have to expand each function $u^{(\alpha)}$ in inverse powers of N :

$$u^{(\alpha)}(n_1, \{m_i\}_{i=1}^K; N) \doteq \sum_{k=1}^{\infty} \frac{1}{N^k} u_k^{(\alpha)}(n_1, \{m_i\}_{i=1}^K).$$

Then \mathcal{P}_ℓ reads

$$\mathcal{P}_\ell = \sum_{\alpha \in \mathbb{Z}} \sum_{i=1}^{\infty} \frac{1}{N^i} \sum_{k=0}^{i-1} L_k u_{i-k}^{(\alpha)}(n_1, \{m_j\}_{j=1}^K) e^{i\alpha(\kappa n - \omega m)}.$$

Performing the multiscale expansion of eq. (3.2) we get several determining equations obtained selecting the different powers of $1/N$ and the different harmonics α .

So, let us write down the resulting determining equations at the lower orders of $1/N$ for the harmonics $\alpha = 0, 1, 2$, necessary to get a dNLS equation as a secularity condition.

The order $1/N$ gives, for $\alpha = 0, 1$, linear equations which are identically satisfied by taking into account the dispersion relation (3.3). For $|\alpha| \geq 2$ one gets some linear equations whose only solution is given by $u_1^{(\alpha)} = 0$.

The order $1/N^2$ gives, for the harmonics $\alpha = 0, 1, 2$, the following equations

$$\left[(\mu + \zeta) M_1 \delta_{n_1} + (\mu - \zeta) \widetilde{M}_1 \delta_{m_1} \right] u_1^{(0)} = 2 \left(-e^{i\kappa} - e^{-i\kappa} + e^{i\omega} + e^{-i\omega} \right) |u_1^{(1)}|^2, \quad (3.4)$$

$$e^{i\kappa} (\mu e^{-i\omega} + \zeta) M_1 \delta_{n_1} u_1^{(1)} + e^{-i\omega} (\mu e^{i\kappa} - \zeta) \widetilde{M}_1 \delta_{m_1} u_1^{(1)} = 0, \quad (3.5)$$

$$\left[\zeta (e^{2i\kappa} - e^{-2i\omega}) + \mu (e^{2i(\kappa-\omega)} - 1) \right] u_2^{(2)} = (-e^{i\kappa} + e^{-i\omega} + e^{i(2\kappa-\omega)} - e^{i(\kappa-2\omega)}) (u_1^{(1)})^2. \quad (3.6)$$

The solution of eq. (3.5) is given by $u_1^{(1)}(n_1, \{m_i\}_{i=1}^K) = u_1^{(1)}(n_2, \{m_i\}_{i=2}^K)$ with $n_2 \doteq n_1 + \gamma m_1$, $\gamma \doteq \mp 1$, provided that the integers M_1 and \widetilde{M}_1 are chosen as

$$M_1 = \gamma S e^{-i\omega} (\mu e^{i\kappa} - \zeta), \quad \widetilde{M}_1 = -S e^{i\kappa} (\mu e^{-i\omega} + \zeta), \quad (3.7)$$

where $S \in \mathbb{C}$ is a constant. As it has been shown in [9] one can always choose $S = r \exp(i\theta)$, with $r > 0$ and $\theta = -\arctan[(\zeta \sin \kappa)/(\zeta \cos \kappa - \mu)]$, in such a way that M_1 and \widetilde{M}_1 are indeed positive integers. Taking into account the dispersion relation (3.3), the coefficients M_1 and \widetilde{M}_1 in eq. (3.7) can be rewritten as

$$M_1 = \gamma S (\mu - \zeta e^{i\kappa}), \quad \widetilde{M}_1 = S e^{i\kappa} \frac{\zeta^2 - \mu^2}{\mu e^{i\kappa} - \zeta}. \quad (3.8)$$

Eqs. (3.4, 3.6) allow to express $u_1^{(0)}$ and $u_2^{(2)}$ in terms of $u_1^{(1)}$ and $\bar{u}_1^{(1)}$. As $u_1^{(1)}$ is a function of n_2 the same must be for $u_1^{(0)}$ and $u_2^{(2)}$, i.e. $u_1^{(0)}(n_1, \{m_i\}_{i=1}^K) = u_1^{(0)}(n_2, \{m_i\}_{i=2}^K)$, $u_2^{(2)}(n_1, \{m_i\}_{i=1}^K) = u_2^{(2)}(n_2, \{m_i\}_{i=2}^K)$. Then, taking into account eq. (3.8), we find that eqs. (3.4, 3.6) reduce respectively to

$$\delta_{n_2} u_1^{(0)} = \alpha_1 |u_1^{(1)}|^2, \quad \alpha_1 \doteq -\frac{2\gamma (1 + e^{i\kappa})^2}{S e^{i\kappa} (\mu + \zeta) (\mu - \zeta e^{i\kappa})}, \quad (3.9)$$

$$u_2^{(2)} = \alpha_2 (u_1^{(1)})^2, \quad \alpha_2 \doteq \frac{1 + e^{i\kappa}}{(1 - e^{i\kappa})(\mu + \zeta)}. \quad (3.10)$$

We can now consider the equation for the harmonic $\alpha = 1$ at order $1/N^3$. We have:

$$\begin{aligned} & (\sigma_1 \delta_{n_1} + \sigma_2 \delta_{m_1}) u_2^{(1)} + (\sigma_3 \delta_{n_1}^2 + \sigma_4 \delta_{m_1}^2 + \sigma_5 \delta_{n_1} \delta_{m_1} + \sigma_6 \delta_{m_2}) u_1^{(1)} = \\ & = u_1^{(1)} (\sigma_7 \delta_{n_1} + \sigma_8 \delta_{m_1}) u_1^{(0)} + \sigma_9 \bar{u}_1^{(1)} u_2^{(2)}, \end{aligned} \quad (3.11)$$

where, taking into account eq. (3.8), the coefficients σ_i , $1 \leq i \leq 9$, read

$$\begin{aligned} \sigma_1 & \doteq \frac{\gamma S e^{i\kappa} (\mu - \zeta e^{i\kappa}) (\mu^2 - \zeta^2)}{\mu e^{i\kappa} - \zeta}, & \sigma_2 & \doteq -\gamma \sigma_1, \\ \sigma_3 & \doteq \frac{1}{2} \gamma S (\mu - \zeta e^{i\kappa}) \sigma_1, & \sigma_4 & \doteq \frac{\sigma_1^2}{2 (\mu - \zeta e^{i\kappa})}, \\ \sigma_5 & \doteq -\frac{\gamma \mu \sigma_1^2}{(\mu^2 - \zeta^2)}, & \sigma_6 & \doteq \widetilde{M}_2 (\mu - \zeta e^{i\kappa}), \\ \sigma_7 & \doteq \frac{\sigma_1 (e^{2i\kappa} - 1)}{e^{i\kappa} (\mu + \zeta)}, & \sigma_8 & \doteq \frac{S e^{i\kappa} (\mu^2 - \zeta^2) (\mu + \zeta) (1 - e^{2i\kappa})}{(\mu e^{i\kappa} - \zeta)^2}, \\ \sigma_9 & \doteq \frac{\zeta \mu (e^{2i\kappa} - 1)^2 (e^{i\kappa} + 1)^2 (\mu - \zeta)}{e^{i\kappa} (\mu - \zeta e^{i\kappa}) (\mu e^{i\kappa} - \zeta)^2}. \end{aligned}$$

Using eqs. (3.9, 3.10) we can write eq. (3.11) as

$$(\sigma_1 \delta_{n_1} + \sigma_2 \delta_{m_1}) u_2^{(1)} = \mathcal{L}(u_1^{(1)}), \quad (3.12)$$

where \mathcal{L} is a linear operator. Notice that the l.h.s. of eq. (3.12) is the same as in formula (3.5), but it involves the field $u_2^{(1)}$, instead of $u_1^{(1)}$. Requiring that no secular term appears we get the following equation for $u_1^{(1)} = u_1^{(1)}(n_2, \{m_i\}_{i=2}^K)$:

$$(\sigma_3 \delta_{n_1}^2 + \sigma_4 \delta_{m_1}^2 + \sigma_5 \delta_{n_1} \delta_{m_1} + \sigma_6 \delta_{m_2}) u_1^{(1)} = u_1^{(1)} (\sigma_7 \delta_{n_1} + \sigma_8 \delta_{m_1}) u_1^{(0)} + \sigma_9 \bar{u}_1^{(1)} u_2^{(2)}. \quad (3.13)$$

Then from eq. (3.12) we see that $u_2^{(1)}$ will satisfy the equation $(\sigma_1 \delta_{n_1} + \sigma_2 \delta_{m_1}) u_2^{(1)} = 0$, i.e. $u_2^{(1)}(n_1, \{m_i\}_{i=1}^K) = u_2^{(1)}(n_2, \{m_i\}_{i=2}^K)$ whenever eq. (3.8) holds. Using eqs. (3.9, 3.10) we find that eq. (3.13) is equivalent to the dNLS equation

$$i \delta_{m_2} u_1^{(1)} = \rho_1 \delta_{n_2}^2 u_1^{(1)} + \rho_2 u_1^{(1)} |u_1^{(1)}|^2, \quad (3.14)$$

where

$$\begin{aligned} \rho_1 &\doteq \frac{i \zeta \mu S^2 e^{i\kappa} (\zeta^2 - \mu^2) (e^{2i\kappa} - 1)}{2 \widetilde{M}_2 (\mu e^{i\kappa} - \zeta)}, \\ \rho_2 &\doteq \frac{i \zeta \mu (\mu - \zeta) (e^{2i\kappa} - 1) (e^{i\kappa} + 1)^4}{\widetilde{M}_2 e^{i\kappa} (\mu + \zeta) (\mu e^{i\kappa} - \zeta)^2 (\zeta e^{i\kappa} - \mu)^2}. \end{aligned}$$

Taking into account the form of S , one finds that ρ_1 and ρ_2 are both real numbers:

$$\begin{aligned} \rho_1 &= \frac{\mu \zeta r^2 (\zeta^2 - \mu^2) \sin \kappa}{\widetilde{M}_2 (\zeta^2 + \mu^2 - 2 \zeta \mu \cos \kappa)}, \\ \rho_2 &= - \frac{8 \zeta \mu (\zeta - \mu) (1 + \cos \kappa)^2 \sin \kappa}{\widetilde{M}_2 (\mu + \zeta) (\zeta^2 + \mu^2 - 2 \zeta \mu \cos \kappa)^2}. \end{aligned}$$

Notice that $\rho_1 \rho_2 < 0$, so that eq. (3.14) is a defocusing dNLS equation.

Eq. (3.14) may be obtained as a compatibility condition of the linear problem

$$\delta_{n_2} \Phi_{n_2, m_2}(\eta) = U(u_1^{(1)}, \bar{u}_1^{(1)}; \eta) \Phi_{n_2, m_2}(\eta), \quad (3.15)$$

$$\delta_{m_2} \Phi_{n_2, m_2}(\eta) = V(u_1^{(1)}, \bar{u}_1^{(1)}; \eta) \Phi_{n_2, m_2}(\eta), \quad (3.16)$$

with

$$\begin{aligned} U(u_1^{(1)}, \bar{u}_1^{(1)}; \eta) &\doteq \begin{pmatrix} i\eta & u_1^{(1)} \\ \bar{u}_1^{(1)} & -i\eta \end{pmatrix}, \\ V(u_1^{(1)}, \bar{u}_1^{(1)}; \eta) &\doteq \begin{pmatrix} 2i\eta^2 + i|u_1^{(1)}|^2 & 2\eta u_1^{(1)} - i\delta_{n_2} u_1^{(1)} \\ 2\eta \bar{u}_1^{(1)} + i\delta_{n_2} \bar{u}_1^{(1)} & -2i\eta^2 - i|u_1^{(1)}|^2 \end{pmatrix}, \end{aligned}$$

being $\eta \in \mathbb{C}$ the spectral parameter and $\Phi_{n_2, m_2}(\eta)$ a discrete complex vector function. A necessary and sufficient condition for the integrability of eq. (3.14) is that the coefficients ρ_1 and ρ_2 be real. Thus eq. (3.14) is integrable iff $\ell = \infty$.

4. CONCLUDING REMARKS

In this paper we have extended the results obtained in [7, 9] to the case of functions of slow-varyness of order infinity. To do so we have used the connection between shift operators and infinite series of differential operators (2.2). Using these formulas we have been able to easily recover the formula (1.1), usually proved by combinatorial techniques [5]. Moreover, in analogy with the continuous case we can introduce partial shift operators, which are expressed in terms of δ operators, namely infinite series of difference operators.

The multiscale expansion of the lpKdV equation in terms of harmonics implies, as a condition for the non existence of secular terms, a NLS equation written in terms of δ operators. Choosing δ_{n_2} in eq. (3.14) as a series of symmetric differences and fixing $\ell = 2$ the resulting dNLS equation is equivalent to the one presented in [7, 9], see eq. (1.2). In this way we have

shown that for any finite order of slow-varyness the obtained dNLS equation will be local and no summation terms will ever appear.

As shown in Section 3, eq. (3.14) has a matrix Lax pair (3.15, 3.16), which is also expressed in terms of δ operators. Such Lax pair has a reduction to any finite order of slow-varyness, provided that the associated wave functions $\Phi_{n_2, m_2}(\eta)$ have the same finite order of slow-varyness. However in the case of finite ℓ the Lax pair (3.15, 3.16) reduces to difference equations and their compatibility is no more satisfied by the slow-varying approximation of eq. (3.14). In fact the difference operators Δ do not satisfy the Leibniz rule.

This work still leaves many open problems on which we are working at the moment. Let us just mention the analysis of the solutions of the lpKdV equation obtained from the exact ones of the reduced equation and their comparison with the ones obtained by carrying out a multiscale expansion of the continuous potential KdV equation; the reduction of symmetries of the lpKdV equation; the reduction of differential-difference equations both integrable and nonintegrable.

It is still an open problem to understand the role played by an integrable discrete equation like the nonlocal discrete NLS equation introduced by Ablowitz and Ladik [1] from the isospectral compatibility of a discrete analogue of the Zakharov-Shabat spectral problem (3.15, 3.16):

$$\begin{aligned}
i \frac{\Delta_m u_{n,m}}{\Delta t} &= \frac{1}{2(\Delta x)^2} \left[\left(u_{n+1,m} - 2u_{n,m} + u_{n-1,m} \prod_{k=-\infty}^{n-1} \Lambda_{k,m} \right) + \right. \\
&+ \left. \left(u_{n+1,m+1} \prod_{k=-\infty}^n \Lambda_{k,m} - 2u_{n,m+1} + u_{n-1,m+1} \right) \right] \pm \\
&\pm \frac{1}{4} \left[u_{n,m} (\bar{u}_{n,m} u_{n+1,m} + \bar{u}_{n,m+1} u_{n+1,m+1}) + \right. \\
&+ u_{n,m+1} (u_{n-1,m} \bar{u}_{n,m} + u_{n-1,m+1} \bar{u}_{n,m+1}) + \\
&+ 2|u_{n,m}|^2 u_{n+1,m+1} \prod_{k=-\infty}^n \Lambda_{k,m} + 2|u_{n,m+1}|^2 u_{n-1,m} \prod_{k=-\infty}^{n-1} \Lambda_{k,m} \Big] - \\
&- u_{n,m} \sum_{k=-\infty}^n \Delta_m S_{k,m} - u_{n,m+1} \sum_{k=-\infty}^{n-1} \Delta_m \bar{S}_{k,m},
\end{aligned} \tag{4.1}$$

with

$$\Lambda_{k,m} \doteq \frac{1 \pm |u_{k,m+1}|^2}{1 \pm |u_{k,m}|^2}, \quad S_{k,m} \doteq u_{k,m} \bar{u}_{k-1,m} + u_{k+1,m} \bar{u}_{k,m}.$$

Eq. (4.1) reduces to the continuous NLS equation when $\Delta t \rightarrow 0$ and $\Delta x \rightarrow 0$

ACKNOWLEDGMENTS

RHH was partially supported by the Region of Madrid and Universidad Polit cnica de Madrid (UPM) through the grant ref. CCG06-UPM/MTM-539. DL and MP were partially supported by PRIN Project SINTESI-2004 of the Italian Minister for Education and Scientific Research. MP was partially supported by the European Community through the FP6 Marie Curie RTN ENIGMA (contract number MRTN-CT-2004-5652).

REFERENCES

- [1] Ablowitz M.J. and Ladik J.F., *Stud. Appl. Math.* **55** (1976) 1011–1018.
- [2] Abramowitz M. and Stegun I.A., *Handbook of mathematical functions with formulas, graphs, and mathematical tables*, Dover Publications, Inc., New York, 1992.
- [3] Agrotis M., Lafortune S. and Kevrekidis P.G., *Discr. Cont. Dyn. Sist.* **2005** supp. 22–29.
- [4] Calogero F. and Eckhaus W., *Inv. Prob.* **3** 2 (1987) 229–262.
Calogero F. and Eckhaus W., *Inv. Prob.* **4** 1 (1987) 11–33.
- [5] Jordan C., *Calculus of finite differences*, R ttig and Romwalter, Sopron, 1939.

- [6] Leon J. and Manna M., *Journ. Phys. A* **32** (1999) 2845-2869.
- [7] Levi D., *Jour. Phys. A* **38** (2005) 7677-7685.
- [8] Levi D. and Heredero R., *Jour. Nonlinear Math. Phys.* **12** 1 (2005) 440-455.
- [9] Levi D. and Petrerera M., *Jour. Math. Phys.* **47** (2006) 043509.
- [10] Levi D. and Petrerera M., *Jour. Phys. A* **40** (2007) 4141-4159.
- [11] Levi D. and Winternitz P., *Jour. Phys. A* **39** (2006) R1-R63.
- [12] Nijhoff F.W. and Capel H.W., *Acta Appl. Math.* **39** (1995) 133-158.
- [13] Papageorgiou V.G., private communication.
- [14] Taniuti T., *Prog. Theor. Phys.* **55** 1 (1974) 1654-1676.
- [15] Taniuti T. and Nishihara K., *Nonlinear waves*, Pitman, Boston, 1983.
- [16] Viallet C., private communication.

UNIVERSIDAD POLITÉCNICA DE MADRID, ESCUELA UNIVERSITARIA DE INGENIERÍA TÉCNICA DE TELECOMUNICACIÓN, DEPARTAMENTO DE MATEMÁTICA APLICADA, CAMPUS SUR CTRA DE VALENCIA KM. 728031, MADRID, SPAIN

E-mail address: rafahh@euitt.upm.es

DIPARTIMENTO DI INGEGNERIA ELETTRONICA, UNIVERSITÀ DEGLI STUDI ROMA TRE AND SEZIONE INFN, ROMA TRE, VIA DELLA VASCA NAVALE 84, 00146 ROMA, ITALY

E-mail address: levi@fis.uniroma3.it

ZENTRUM MATHEMATIK, TECHNISCHE UNIVERSITÄT MÜNCHEN, BOLTZMANNSTR. 3, D-85747 GARCHING BEI MÜNCHEN, GERMANY, AND DIPARTIMENTO DI INGEGNERIA ELETTRONICA, UNIVERSITÀ DEGLI STUDI ROMA TRE, VIA DELLA VASCA NAVALE 84, 00146 ROMA, ITALY

E-mail address: petrera@ma.tum.de

DIPARTIMENTO DI FISICA E DIPARTIMENTO DI INGEGNERIA ELETTRONICA, UNIVERSITÀ DEGLI STUDI ROMA TRE AND SEZIONE INFN, ROMA TRE, VIA DELLA VASCA NAVALE 84, 00146 ROMA, ITALY

E-mail address: scimiterna@fis.uniroma3.it